

STRONG APPROXIMATION BY MARCINKIEWICZ MEANS OF TWO-DIMENSIONAL WALSH-KACZMARZ-FOURIER SERIES

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ABSTRACT. In this paper we study the exponential uniform strong approximation of Marcinkiewicz type of two-dimensional Walsh-Kaczmarz-Fourier series. In particular, it is proved that the Marcinkiewicz type of two-dimensional Walsh-Kaczmarz-Fourier series of the continuous function f is uniformly strong summable to the function f exponentially in the power $1/2$. Moreover, it is proved that this result is best possible.

It is known that there exist continuous functions the trigonometric (Walsh) Fourier series of which do not converge uniformly. However, as it was proved by Fejér [2] in 1904, the arithmetic means of the differences between the function and its Fourier partial sums converge uniformly to zero. The problem of strong summation was initiated by Hardy and Littlewood [17]. They generalized Fejér's result by showing that the strong means also converge uniformly to zero for any continuous function. The investigation of the rate of convergence of the strong means was started by Alexits [1]. Many papers have been published which are closely related with strong approximation and summability. We note that a number of significant results are due to Leindler [19, 20, 21], Totik [29, 30, 31], Fridli and Schipp [5], Gogoladze [10], Goginava, Gogoladze, Karagulyan [14]. Leindler has also published a monograph [22].

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system see Schipp [24, 25, 26], Fridli, Schipp [4, 5], Fridli [3], Rodin [23], Goginava, Gogoladze [13, 12], Gát, Goginava, Karagulyan [6, 7], Goginava, Gogoladze, Karagulyan [14] and concerning the Ciesielski system see Weisz [32, 33]. The summability of multiple Walsh-Fourier series have been investigated in [8, 15, 16, 18, 34].

Fridli [3] proved that the following theorem is true.

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Theorem F. *Let ψ be monotonically increasing function defined on $[0, \infty)$ for which $\lim_{u \rightarrow 0+} \psi(u) = 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi(|S_k^\kappa(f; x) - f(x)|) = 0 \quad (f \in C(G))$$

if and only if there exists $A > 0$ such that $\psi(t) \leq \exp(At)$ ($0 \leq t < \infty$). Moreover, the convergence is uniform in x .

In this paper we study the exponential uniform strong approximation of the Marcinkiewicz means of the two-dimensional Walsh-Kaczmarz-Fourier series. In particular, it is proved that the Marcinkiewicz type of the two-dimensional Walsh-Kaczmarz-Fourier series of the continuous function f is uniformly strong summable to the function f exponentially in the power $1/2$. Moreover, it is proved that this result is best possible.

1. WALSH FUNCTIONS

Let \mathbb{P} denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote \mathbb{Z}_2 the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on \mathbb{Z}_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups \mathbb{Z}_2 . The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with coordinates $x_k \in \{0, 1\}$ ($k \in \mathbb{N}$). The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighbourhoods of G can be given in the following way [27]:

$$I_0(x) := G,$$

$I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$
 $(x \in G, n \in \mathbb{N})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G , $I_n := I_n(0)$ ($n \in \mathbb{N}$). Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, the n th coordinate of which is 1 and the rest are zeros ($n \in \mathbb{N}$).

For $k \in \mathbb{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the k th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\}$ ($i \in \mathbb{N}$), i. e. n is expressed in the number system of base 2. Let us denote the order of n by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k}.$$

For $A \in \mathbb{N}$ define the transformation $\tau_A: G \rightarrow G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_0, x_A, x_{A+1}, \dots).$$

By the definition of τ_A (see [28]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2|n|}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^\alpha(x) := \sum_{k=0}^{n-1} \alpha_k(x), \quad (n \in \mathbb{N}),$$

where $\alpha_k = w_k$ (for all $k \in \mathbb{P}$) or κ_k (for all $k \in \mathbb{P}$). Recall that (see [27])

$$(1) \quad D_{2^n}(x) := D_{2^n}^w(x) = D_{2^n}^\kappa(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases}$$

$$(2) \quad D_n^w(t) = w_n(t) \sum_{j=0}^{\infty} n_j w_{2^j}(t) D_{2^j}(t),$$

where $n = \sum_{j=0}^{\infty} n_j 2^j$. The k th partial sum of the Walsh(-Kaczmarz)-Fourier series of function f at point x is denoted by $S_k^\alpha(f; x)$.

The Fejér kernels are defined as follows

$$K_n^\alpha(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha(x).$$

The Kronecker product $(\alpha_{n,m} : n, m \in \mathbb{N})$ of two Walsh(-Kaczmarz) system is said to be the two-dimensional Walsh(-Kaczmarz) system. Thus,

$$\alpha_{n,m}(x, y) = \alpha_n(x) \alpha_m(y).$$

If $f \in L_1(G^2)$, then the number $\hat{f}^\alpha(n, m) := \int_{G^2} f \alpha_{n,m}$ ($n, m \in \mathbb{N}$) is said to be the (n, m) th Walsh(-Kaczmarz)-Fourier coefficient of f . Denote by $S_{n,m}^\alpha$ the (n, m) th partial sum of the Walsh(-Kaczmarz)-Fourier series of a function f . Namely,

$$S_{n,m}^\alpha(f; x, y) := \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \hat{f}^\alpha(k, i) \alpha_{k,i}(x, y).$$

Let us fix $d \geq 1, d \in \mathbb{P}$. For Walsh group G let G^d be its Cartesian product $G \times \dots \times G$ taken with itself d -times.

The norm (or quasinorm) of the space L_p is defined by

$$\|f\|_p := \left(\int_{G^2} |f(x, y)|^p d\mu(x, y) \right)^{1/p} \quad (0 < p < +\infty).$$

2. BEST APPROXIMATION

Denote by $E_{l,r}(f)$ the best approximation of a function $f \in C(G^2)$ by Walsh-Kaczmarz polynomials of degree $\leq l$ of a variable x and of degree $\leq r$ of a variable y and let $E_l^{(1)}(f)$ be the partial best approximation of a function $f \in C(G^2)$ by Walsh-Kaczmarz polynomials of degree $\leq l$ of a variable x , whose coefficients are continuous functions of the remaining variable y , in particular, best approximation with respect to polynomials $T_l^{(1)}(x, y) := \sum_{j=0}^{l-1} \alpha_j(y) \kappa_j(x)$. Analogously, we can define $E_r^{(2)}(f)$.

Let $2^L \leq l < 2^{L+1}$ and $E_{2^L, 2^L}(f) := \|f - T_{2^L, 2^L}\|_C$, where $E_{2^L, 2^L}(f)$ is the best approximation of $f \in C(G^2)$ by Walsh-Kaczmarz polynomials $T_{2^L, 2^L}$.

Since

$$\|S_{2^L, 2^L}(f)\|_C \leq \|f\|_C$$

we can write

$$\begin{aligned} |S_{l,l}^\kappa(f; x, y) - f(x, y)| &\leq |S_{l,l}^\kappa(f - S_{2^L, 2^L}(f); x, y)| + \|S_{2^L, 2^L}(f) - f\|_C \\ &\leq |S_{l,l}^\kappa(f - S_{2^L, 2^L}(f); x, y)| + \|S_{2^L, 2^L}(f - T_{2^L, 2^L}(f))\|_C \\ &\quad + \|f - T_{2^L, 2^L}(f)\|_C \\ &\leq |S_{l,l}^\kappa(f - S_{2^L, 2^L}(f); x, y)| + 2E_{2^L, 2^L}(f). \end{aligned}$$

It is well known that (see [12])

$$(3) \quad E_{2^L, 2^L}(f) \leq 2E_{2^L}^{(1)}(f) + 2E_{2^L}^{(2)}(f).$$

It is easily seen that

$$(4) \quad \|f - S_{2^L, 2^L}(f)\|_C \leq 2E_{2^L, 2^L}(f).$$

3. MAIN RESULTS

Theorem 1. *Let $f \in C(G^2)$. Then there exists a positive constant $c(f, A)$ depending only on f and A such that the inequality*

$$\begin{aligned} &\left\| \frac{1}{n} \sum_{l=1}^n \left(e^{A|S_{l,l}^\kappa(f) - f|^{1/2}} - 1 \right) \right\|_C \\ &\leq \frac{c(f, A)}{n} \sum_{l=1}^n \left(\sqrt{E_l^{(1)}(f)} + \sqrt{E_l^{(2)}(f)} \right) \end{aligned}$$

is satisfied for any $A > 0$.

We say that the function ψ belongs to the class Ψ if it increases on $[0, +\infty)$ and

$$\lim_{u \rightarrow 0} \psi(u) = \psi(0) = 0.$$

Theorem 2. a) Let $\varphi \in \Psi$ and let the inequality

$$(5) \quad \overline{\lim}_{u \rightarrow \infty} \frac{\varphi(u)}{\sqrt{u}} < \infty$$

hold. Then for any function $f \in C(G^2)$ the equality

$$(6) \quad \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{l=1}^n \left(e^{\varphi(|S_{li}^k(f) - f|)} - 1 \right) \right\|_C = 0$$

is satisfied.

b) For any function $\varphi \in \Psi$ satisfying the condition

$$(7) \quad \overline{\lim}_{u \rightarrow \infty} \frac{\varphi(u)}{\sqrt{u}} = \infty$$

there exists a function $F \in C(G^2)$ such that

$$\overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \sum_{l=1}^m \left(e^{\varphi(|S_{li}^k(F; 0, 0) - f(0, 0)|)} - 1 \right) = +\infty.$$

4. AUXILIARY RESULTS

In this paper c is a positive constant, which is not necessary the same at different occurrences.

Lemma 1. (Gogoladze [10]) Let $\varphi, \psi \in \Psi$ and the equality

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n \psi(|S_{li}^\kappa(f; x, y) - f(x, y)|) = 0$$

be satisfied at the point (x_0, y_0) or uniformly on a set $E \subset G^2$. If

$$\overline{\lim}_{u \rightarrow \infty} \frac{\varphi(u)}{\psi(u)} < \infty,$$

then the equality

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n \varphi(|S_{li}^\kappa(f; x, y) - f(x, y)|) = 0$$

is satisfied at the point (x_0, y_0) or uniformly on a set $E \subset G^2$.

Moreover, we will use the next Lemma of Glukhov [9, p. 670].

Lemma 2 (Glukhov [9]). *Let $\alpha_1, \dots, \alpha_n$ be real numbers. Let $p \in \mathbb{P}$ and $1 < q \leq 2$. Then*

$$\frac{1}{n} \int_{G^p} \left| \sum_{l=1}^n \alpha_l \prod_{k=1}^p D_l^w(x_k) \right| d\mu(x_1, \dots, x_p) \leq \frac{c}{n^{1/q}} \left(\sum_{k=1}^n |\alpha_k|^q \right)^{1/q},$$

where c is depend only on p and q .

In paper [9, p. 672, l. 12-13] it is stated that constant c depend on dimension and in dimension p it will be $cp!$. Now, we choose α_k as special numbers in Lemma of Glukhov. Set

$$\alpha_k = \begin{cases} 1, & k = 2^{n-1}, \dots, 2^n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

We immediately have

Corollary 1. *Let $p \in \mathbb{P}$. Then there exists an absolute constant c such that*

$$(8) \quad \sup_n \int_{G^p} \frac{1}{2^n} \left| \sum_{l=2^{n-1}}^{2^n-1} \prod_{k=1}^p D_l^w(x_k) \right| d\mu(x_1, \dots, x_p) \leq cp!.$$

Lemma 3. *There exists an absolute constant c such that the inequality*

$$(9) \quad \sup_n \int_{G^d} \frac{1}{2^n} \left| \sum_{j=2^{n-1}}^{2^n-1} \prod_{k=1}^d D_j^\kappa(x_k) \right| d\mu(x_1, \dots, x_p) \leq cd!2^d$$

holds.

Proof. It is known (see Skvortsov [28]) that

$$D_{2^A+j}^\kappa(x) = D_{2^A}(x) + r_A(x)D_j^w(\tau_A(x)), \quad 0 \leq j < 2^A.$$

This implies

$$\begin{aligned} \prod_{k=1}^d D_{2^{n-1}+j}^\kappa(x_k) &= \prod_{k=1}^d (D_{2^{n-1}}(x_k) + r_{n-1}(x_k)D_j^w(\tau_{n-1}(x_k))) \\ &= \sum_{l=0}^d \sum_{\substack{k_1, \dots, k_l \in \{1, \dots, d\} \\ k_r \neq k_s \text{ if } r \neq s}} \prod_{m=1}^l D_{2^{n-1}}(x_{k_m}) \prod_{\substack{k'_q \in S_d^l}} r_{n-1}(x_{k'_q}) D_j^w(\tau_{n-1}(x_{k'_q})) \end{aligned}$$

with the notation $S_d^l := \{1, \dots, d\} \setminus \{k_1, \dots, k_l\}$. That is, we have

$$\left| \sum_{j=0}^{2^{n-1}-1} \prod_{k=1}^d D_{2^{n-1}+j}^\kappa(x_k) \right| \leq$$

$$\begin{aligned} &\leq \sum_{l=0}^d \sum_{\substack{k_1, \dots, k_l \in \{1, \dots, d\} \\ k_r \neq k_s \text{ if } r \neq s}} \left| \prod_{m=1}^l D_{2^{n-1}}(x_{k_m}) \prod_{k'_q \in S_d^l} r_{n-1}(x_{k'_q}) \right. \\ &\quad \times \left. \sum_{j=0}^{2^{n-1}-1} \prod_{k'_q \in S_d^l} D_j^w(\tau_{n-1}(x_{k'_q})) \right| \end{aligned}$$

and

$$\begin{aligned} L_n &:= \int_{G^d} \frac{1}{2^n} \left| \sum_{j=2^{n-1}}^{2^n-1} \prod_{k=1}^d D_j^\kappa(x_k) \right| d\mu(x_1) \dots d\mu(x_d) \\ &\leq \sum_{l=0}^d \sum_{\substack{k_1, \dots, k_l \in \{1, \dots, d\} \\ k_r \neq k_s \text{ if } r \neq s}} \int_{G^{d-l}} \frac{1}{2^n} \left| \sum_{j=0}^{2^{n-1}-1} \prod_{k'_q \in S_d^l} D_j^w(\tau_{n-1}(x_{k'_q})) \right| d\mu(x_{k'_1}) \dots d\mu(x_{k'_{d-l}}). \end{aligned}$$

Since the transformation $\tau_{n-1}: G \rightarrow G$ is measure-preserving [28] and inequality 8 immediately yields

$$\begin{aligned} L_n &\leq \sum_{l=0}^d \sum_{\substack{k_1, \dots, k_l \in \{1, \dots, d\} \\ k_r \neq k_s \text{ if } r \neq s}} \int_{G^{d-l}} \frac{1}{2^n} \left| \sum_{j=0}^{2^{n-1}-1} \prod_{k'_q \in S_d^l} D_j^w(x_{k'_q}) \right| d\mu(x_{k'_1}) \dots d\mu(x_{k'_{d-l}}) \\ &\leq c \sum_{l=0}^d \sum_{\substack{k_1, \dots, k_l \in \{1, \dots, d\} \\ k_r \neq k_s \text{ if } r \neq s}} (d-l)! \leq d! \sum_{l=0}^d \sum_{\substack{k_1, \dots, k_l \in \{1, \dots, d\} \\ k_r \neq k_s \text{ if } r \neq s}} 1. \end{aligned}$$

Since, the number of all subsets of the set $\{1, \dots, d\}$ is 2^d , we immediately have

$$L_n \leq cd!2^d.$$

Taking the supremum for all $n \in \mathbb{N}$ completes the proof of Lemma 3. \square

Lemma 4. *Let $p > 0$. Then*

$$(10) \quad \left\{ \frac{1}{2^A} \sum_{l=2^A}^{2^{A+1}-1} |S_{l,t}^\kappa(f; x, y)|^p \right\}^{1/p} \leq c \|f\|_C (p+1)^2.$$

Proof. Since

$$\left\{ \frac{1}{2^A} \sum_{l=2^A}^{2^{A+1}-1} |S_{l,l}^\kappa(f; x, y)|^p \right\}^{1/p} \leq \left\{ \frac{1}{2^A} \sum_{l=2^A}^{2^{A+1}-1} |S_{l,l}^\kappa(f; x, y)|^{p+1} \right\}^{1/(p+1)}$$

without loss of generality we can suppose that $p = 2^m, m \in \mathbb{P}$. We can write

$$\begin{aligned} |S_{l,l}^\kappa(f; x, y)|^2 &= S_{l,l}^\kappa(f; x, y) S_{l,l}^\kappa(f; x, y) \\ &= \int_{G^2} f(x + s_1, y + t_1) D_l^\kappa(s_1) D_l^\kappa(t_1) d\mu(s_1, t_1) \\ &\quad \times \int_{G^2} f(x + s_2, y + t_2) D_l^\kappa(s_2) D_l^\kappa(t_2) d\mu(s_2, t_2) \\ &= \int_{G^4} f(x + s_1, y + t_1) f(x + s_2, y + t_2) \\ &\quad \times D_l^\kappa(s_1) D_l^\kappa(s_2) D_l^\kappa(t_1) D_l^\kappa(t_2) d\mu(s_1, t_1, s_2, t_2). \end{aligned}$$

Hence from Lemma 3, we get

$$\begin{aligned} |S_{l,l}^\kappa(f; x, y)|^p &= \left(|S_{l,l}^\kappa(f; x, y)|^2 \right)^{p/2} \\ &= \int_{G^{2p}} \prod_{k=1}^p f(x + s_k, y + t_k) \prod_{i=1}^p D_l^\kappa(s_i) \prod_{j=1}^p D_l^\kappa(t_j) d\mu(s_1, t_1, \dots, s_p, t_p), \end{aligned}$$

and

$$\begin{aligned} &\left\{ \frac{1}{2^A} \sum_{l=2^A}^{2^{A+1}-1} |S_{l,l}^\kappa(f; x, y)|^p \right\}^{1/p} \\ &\leq \left(\int_{G^{2p}} \prod_{k=1}^p f(x + s_k, y + t_k) \right. \\ &\quad \times \frac{1}{2^A} \left| \sum_{l=2^A}^{2^{A+1}-1} \prod_{i=1}^p D_l^\kappa(s_i) \prod_{j=1}^p D_l^\kappa(t_j) \right| d\mu(s_1, t_1, \dots, s_p, t_p) \Big)^{1/p} \\ &\leq \|f\|_C \left(\int_{G^{2p}} \frac{1}{2^A} \left| \sum_{l=2^A}^{2^{A+1}-1} \prod_{i=1}^p D_l^\kappa(s_i) \prod_{j=1}^p D_l^\kappa(t_j) \right| d\mu(s_1, t_1, \dots, s_p, t_p) \right)^{1/p} \\ &\leq cp^2 \|f\|_C. \end{aligned}$$

Lemma 4 is proved. \square

Lemma 5. *Let $f \in C(G^2)$ and $p > 0$. Then*

$$(11) \quad \begin{aligned} & \frac{1}{n} \sum_{l=1}^n |S_{l,l}^\kappa(f; x, y) - f(x, y)|^p \\ & \leq c^p \cdot (p+1)^{2p} \left\{ \frac{1}{n} \sum_{l=1}^n \left(E_l^{(1)}(f) \right)^p + \frac{1}{n} \sum_{r=1}^n \left(E_r^{(2)}(f) \right)^p \right\}. \end{aligned}$$

Proof. Since

$$(a+b)^\beta \leq 2^\beta (a^\beta + b^\beta), \beta > 0$$

using (3)-(4) and Lemma 4 we get

$$(12) \quad \begin{aligned} & \frac{1}{2^A} \sum_{l=2^A}^{2^{A+1}-1} |S_{l,l}^\kappa(f; x, y) - f(x, y)|^p \\ & \leq \frac{2^p}{2^A} \sum_{l=2^A}^{2^{A+1}-1} |S_{l,l}^\kappa(f - S_{2^A, 2^A}(f); x, y)|^p + 2^{2p} E_{2^A, 2^A}^p(f) \\ & \leq c^p (p+1)^{2p} \|f - S_{2^A, 2^A}(f)\|_C^p + c^p E_{2^A, 2^A}^p(f) \\ & \leq c^p (p+1)^{2p} \left(\left(E_{2^A}^{(1)}(f) \right)^p + \left(E_{2^A}^{(2)}(f) \right)^p \right). \end{aligned}$$

Let $2^N \leq n < 2^{N+1}$. Then from (12) we have

$$\begin{aligned} & \frac{1}{n} \sum_{l=1}^n |S_{l,l}^\kappa(f; x, y) - f(x, y)|^p \\ & \leq \frac{1}{n} \sum_{l=1}^{2^{N+1}-1} |S_{l,l}^\kappa(f; x, y) - f(x, y)|^p \\ & = \frac{1}{n} \sum_{A=0}^N \sum_{l=2^A}^{2^{A+1}-1} |S_{l,l}^\kappa(f; x, y) - f(x, y)|^p \\ & \leq \frac{c^p (p+1)^{2p}}{n} \sum_{A=0}^N 2^A \left(\left(E_{2^A}^{(1)}(f) \right)^p + \left(E_{2^A}^{(2)}(f) \right)^p \right) \\ & \leq \frac{c^p (p+1)^{2p}}{n} \sum_{A=1}^N \sum_{l=2^{A-1}}^{2^A-1} \left(\left(E_l^{(1)}(f) \right)^p + \left(E_l^{(2)}(f) \right)^p \right) \\ & \leq c^p \cdot (p+1)^{2p} \left\{ \frac{1}{n} \sum_{l=1}^n \left(E_l^{(1)}(f) \right)^p + \frac{1}{n} \sum_{r=1}^n \left(E_r^{(2)}(f) \right)^p \right\}. \end{aligned}$$

Lemma 5 is proved. \square

5. PROOFS OF MAIN RESULTS

The Walsh-Paley version of Theorem 1 were proved in [12]. Based on inequality (11) the same construction works for the Walsh-Kaczmarz case. Therefore the proof of Theorem 1 will be omitted.

Proof of Theorem 2. a) It is easily seen that if $\varphi \in \Psi$, then $e^\varphi - 1 \in \Psi$. Besides, (5) implies the existence of a number A such that

$$\overline{\lim}_{u \rightarrow \infty} \frac{e^{\varphi(u)} - 1}{e^{Au^{1/2}} - 1} < \infty.$$

Therefore, in view of Lemma 1, to prove Theorem 2 it is sufficient to show that

$$(13) \quad \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{l=1}^n \left(e^{A|S_{l,i}^\kappa(f) - f|^{1/2}} - 1 \right) \right\|_C = 0.$$

The validity of equality (13) immediately follows from Theorem 1.

b) Such a construction for the analogical problem has already been made for the Walsh-Paley case [12], where Walsh-Paley function defined on $[0,1]$. We will use idea from above mentioned paper and we construct similar, but not the same function for Walsh-Kaczmarz case, where Walsh-Kaczmarz system is defined on Walsh group. The common aspect of two construction is stated in the inequality (23), later.

First of all, let us prove the validity of point b) in the one-dimensional case. In particular, we prove that if $\psi \in \Psi$ satisfying the condition

$$\overline{\lim}_{u \rightarrow \infty} \frac{\psi(u)}{u} = \infty,$$

then there exists a function $f \in C(G)$ such that

$$(14) \quad \overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \sum_{l=1}^m \left(e^{\psi(|S_l^\kappa(f;0) - f(0)|)} \right) = +\infty.$$

Let $\{B_k : k \geq 1\}$ be an increasing sequence of positive integers such that

$$(15) \quad B_k > 2B_{k-1},$$

$$(16) \quad \frac{\psi(B_k)}{B_k} > \frac{5k}{c'},$$

where c' will be defined later.

Set

$$A_k := \left\lceil \frac{k}{c'} B_k \right\rceil$$

and

$$N_{A_k} := 2^{2A_k} + 2^{2A_k-2} + \dots + 2^2 + 2^0,$$

Set

$$f_j(x) := \frac{1}{j+1} \sum_{l=A_{j-1}}^{A_j-1} \sum_{x_0=0}^1 \cdots \sum_{x_{2A_j-2l-1}=0}^1 \operatorname{sgn} \left(D_{N_{A_j}}^{\kappa}(x) \right) \\ \times \mathbb{I}_{I_{2A_j+2}}(x_0, \dots, x_{2A_j-2l-1}, x_{2A_j-2l}=1, 0, \dots, 0)(x), \\ f(x) := \sum_{j=0}^{\infty} f_j(x), \quad f(0) = 0,$$

where \mathbb{I}_E is characteristic function of the set $E \subset G_m$.

It is easily seen that $f \in C(G)$.

We can write

$$(17) \quad \left| S_{N_{A_k}}^{\kappa}(f; 0) - f(0) \right| = \left| S_{N_{A_k}}^{\kappa}(f; 0) \right| \\ = \left| \int_G f(t) D_{N_{A_k}}^{\kappa}(t) d\mu(t) \right| \\ \geq \left| \int_G f_k(t) D_{N_{A_k}}^{\kappa}(t) d\mu(t) \right| \\ - \sum_{j=k+1}^{\infty} \left| \int_G f_j(t) D_{N_{A_k}}^{\kappa}(t) d\mu(t) \right| \\ - \sum_{j=0}^{k-1} \left| \int_G f_j(t) D_{N_{A_k}}^{\kappa}(t) d\mu(t) \right| \\ = J_1 - J_2 - J_3.$$

From the definition of the function f we have

$$(18) \quad J_1 = \frac{1}{k+1} \sum_{l=A_{k-1}}^{A_k-1} \sum_{t_0=0}^1 \cdots \sum_{t_{2A_k-2l-1}=0}^1 \\ \int \left| D_{N_{A_k}}^{\kappa}(t) \right| d\mu(t) \\ I_{2A_k+2}(t_0, \dots, t_{2A_k-2l-1}, t_{2A_k-2l}=1, 0, \dots, 0)$$

Since (see Skvortsov [28])

$$(19) \quad D_{N_{A_k}}^{\kappa}(t) = D_{2^{2A_k}}^w(t) + r_{2A_k}(t) D_{N_{A_k-1}}^w(\tau_{2A_k}(t)) \\ = r_{2A_k}(t) D_{N_{A_k-1}}^w(\tau_{2A_k}(t)),$$

we can write

$$\left| D_{N_{A_k}}^{\kappa}(t) \right| = \left| D_{N_{A_k-1}}^w(\tau_{2A_k}(t)) \right|$$

$$\begin{aligned}
&= \left| \sum_{j=0}^l r_{2j}(\tau_{2A_k}(t)) D_{2j}(\tau_{2A_k}(t)) \right| \\
&\geq 2^{2l} - \sum_{j=0}^{l-1} 2^{2j} \geq c 2^{2l},
\end{aligned}$$

$$t \in I_{2A_k+2}(t_0, \dots, t_{2A_k-2l-1}, t_{2A_k-2l} = 1, 0, \dots, 0).$$

Hence from (15) and (18) we have

$$\begin{aligned}
(20) \quad J_1 &\geq \frac{c}{k} \sum_{l=A_{k-1}}^{A_k-1} \sum_{t_0=0}^1 \cdots \sum_{t_{2A_k-2l-1}=0}^1 \frac{2^{2l}}{2^{2A_k}} \\
&\geq \frac{c}{k} \sum_{l=A_{k-1}}^{A_k-1} \frac{2^{2l} 2^{2A_k-2l}}{2^{2A_k}} \geq \frac{c(A_k - A_{k-1})}{k} \geq \frac{c_0 A_k}{k}.
\end{aligned}$$

For J_2 we have

$$\begin{aligned}
(21) \quad J_2 &\leq c \sum_{j=k+1}^{\infty} \frac{1}{j+1} \sum_{l=A_{j-1}}^{A_j-1} \frac{2^{2A_j-2l}}{2^{2A_j}} N_{A_k} \\
&\leq \frac{c N_{A_k}}{k} \sum_{l=A_k}^{\infty} \frac{1}{2^l} \leq \frac{c}{k}.
\end{aligned}$$

By (1) and from the construction of the function f_j we can write

$$\begin{aligned}
&\int_G f_j(t) D_{N_{A_k}}^{\kappa}(t) d\mu(t) \\
&= \int_G f_j(t) r_{2A_k}(t) D_{N_{A_k-1}}^w(\tau_{2A_k}(t)) d\mu(t) = 0, \quad j = 1, 2, \dots, k-1
\end{aligned}$$

consequently

$$(22) \quad J_3 = 0.$$

Combining (16)-(22) we conclude that

$$\begin{aligned}
(23) \quad \left| S_{N_{A_k}}^{\kappa}(f; 0) \right| &= \left| S_{N_{A_k}}^{\kappa}(f; 0) - f(0) \right| \geq \frac{c' A_k}{k} = B_k, \\
\psi \left(\left| S_{N_{A_k}}^{\kappa}(f; 0) \right| \right) &\geq \psi(B_k) \geq \frac{5k}{c'} B_k \geq 5A_k.
\end{aligned}$$

We note that for Walsh-Fourier series function with properties (23) was constructed in [12]. The construction in [12] is given for $[0, 1)$ interval.

Let us write $\varphi(u) = \lambda(u) \sqrt{u}$ and define $\psi(u) := \lambda(u^2) u$. Then

$$\lim_{u \rightarrow \infty} \frac{\psi(u)}{u} = +\infty.$$

Therefore (see (23)) there exists a function $f \in C(G)$ for which

$$(24) \quad \psi \left(\left| S_{N_{A_k}}^\kappa (f, 0) \right| \right) \geq 5A_k.$$

Set

$$F(x, y) := f(x) f(y).$$

It is easy to show that

$$\begin{aligned} \varphi \left(\left| S_{N_{A_k}, N_{A_k}}^\kappa (F; 0, 0) \right| \right) &= \varphi \left(\left| S_{N_{A_k}}^\kappa (f; 0) \right|^2 \right) \\ &= \lambda \left(\left| S_{N_{A_k}}^\kappa (f; 0) \right|^2 \right) \left| S_{N_{A_k}}^\kappa (f; 0) \right| \\ &= \psi \left(\left| S_{N_{A_k}}^\kappa (f; 0) \right| \right). \end{aligned}$$

Consequently, from (24) we have

$$\begin{aligned} \frac{1}{N_{A_k}} \sum_{i=1}^{N_{A_k}} e^{\varphi(|S_{i,i}^\kappa(F;0,0)|)} &\geq \frac{1}{N_{A_k}} e^{\varphi \left(\left| S_{N_{A_k}, N_{A_k}}^\kappa (F; 0, 0) \right| \right)} \\ &= \frac{1}{N_{A_k}} e^{\psi \left(\left| S_{N_{A_k}}^\kappa (f; 0) \right| \right)} \\ &\geq \frac{e^{5A_k}}{2^{2A_k}} \rightarrow \infty \text{ as } k \rightarrow \infty. \end{aligned}$$

Theorem 2 is proved. \square

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